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Diffusion escape through a cluster of small absorbing windows

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Abstract
We study the first eigenvalue of the Laplace equation in a bounded domain in $\mathbb{R}^d (d = 2, 3)$ with mixed Neumann–Dirichlet (Zaremba) boundary conditions. The Neumann condition is imposed on most of the boundary and the Dirichlet boundary consists of a cluster of small windows. When the windows are well separated the first eigenvalue is asymptotically the sum of eigenvalues of mixed problems with a single Dirichlet window. However, when two or more Dirichlet windows cluster tightly together they interact nonlinearly. We compare our asymptotic approximation of the eigenvalue to the escape rate of simulated Brownian particles through the small windows.

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1. Introduction

The first eigenvalue $\lambda_1(\varepsilon)$ of the Laplace equation in a bounded domain Ω, with mixed Neumann–Dirichlet boundary conditions, decreases to zero as the Dirichlet part of the boundary, $\partial\Omega_a$, shrinks and disappears. The reciprocal of the first eigenvalue in this limit is asymptotically the mean first passage time (MFPT) of Brownian motion to $\partial\Omega_a$, if its trajectories are reflected at $\partial\Omega_r = \partial\Omega - \partial\Omega_a$. This problem has been studied in various geometries, mostly under the assumption that $\partial\Omega_a$ consists of well-separated circular windows (for $d = 3$) or arcs (for $d = 2$), analytically in [1–8] and numerically in [9] for a sphere in $\mathbb{R}^3$. It was shown that for the case of a single Dirichlet circular window of radius a (in three dimensions, $|\partial\Omega_a| = \pi a^2$) and a single Dirichlet arc (in two dimensions) the MFPT (the reciprocal of the first eigenvalue) is given by

$$\frac{1}{\lambda_1(\varepsilon)} \sim \bar{\tau}_\varepsilon = \begin{cases} \dfrac{|\Omega|}{\pi D}\log\dfrac{1}{\varepsilon}(1+o(1)) & \text{for} \quad d=2 \\ \dfrac{|\Omega|}{4aD}(1+o(1)) & \text{for} \quad d=3, \end{cases} \tag{1}$$

where

$$\varepsilon = \left(\frac{|\partial\Omega_a|}{|\partial\Omega|}\right)^{1/(d-1)} \ll 1.$$

Note that for $d = 2$ equation (1) holds for any absorbing arc of length $\alpha\varepsilon$, where $\alpha = O(1)$ for $\varepsilon \ll 1$. If there are several well-separated Dirichlet windows, A_i, $(i = 1, \ldots, n)$, then

$$\lambda_1(\varepsilon) \sim \frac{1}{\bar{\tau}_\varepsilon} = \sum_{i=1}^{n} \frac{1}{\bar{\tau}_{\varepsilon_i}}, \tag{2}$$

where $\bar{\tau}_{\varepsilon_i}$ is given by the single-window result (1) with $\partial\Omega_a = A_i$.

In this paper, we study analytically the mixed problem for a Dirichlet boundary $\partial\Omega_a$ that consists of a collection of small windows embedded in an otherwise Neumann boundary $\partial\Omega_r$ of a regular bounded domain Ω. Equivalently, we study the MFPT of Brownian motion to $\partial\Omega_a$. In particular, we show that when the small Dirichlet windows form a cluster, the MFPT to one is influenced by the others, which is not the case for well-separated windows. We generalize the method of [5–8], which consists in deriving and solving Helmholtz's integral equation on the boundary, to the case of several separated or clustered windows. The method of [1–4] applies in a straightforward manner to the case of well-separated windows, where the leading order boundary layer problem is that of the electrified disk, and was solved explicitly by Weber in 1873 [10, 11], but for clustered windows the leading order boundary layer problem is that of two, or more electrified disks. The asymptotic solution of the Helmholtz integral equation circumvents this difficulty and reveals the nonlinear interaction between the clustered windows. We find the explicit dependence of the MFPT on the distance between the windows and demonstrate that the result (1) is recovered as the windows drift apart and a new result is obtained as the windows touch (for $d = 3$) or merge (for $d = 2$). Specifically, for $d = 2$ and a regular domain with two Dirichlet arcs of lengths 2ε and 2δ (normalized by the perimeter $|\partial\Omega|$) and separated by the Euclidean distance $\Delta = \varepsilon + \Delta' + \delta$ between their centers, and for two Dirichlet circular windows of small radii a and b, separated by the Euclidean distance $\Delta = a + \Delta' + b$ between their centers (see figure 1), we obtain

$$\frac{1}{\lambda_1(\varepsilon,\delta)} \sim \bar{\tau}_\varepsilon = \begin{cases} \dfrac{|\Omega|}{\pi D\left(\log\frac{1}{\varepsilon} + \log\frac{1}{\delta}\right)} \dfrac{\log\frac{1}{\delta}\log\frac{1}{\varepsilon} - [\log|\varepsilon+\Delta'+\delta| + O(1)]^2}{1 + 2\frac{\log|\varepsilon+\Delta'+\delta| + O(1)}{\log\frac{1}{\delta}+\log\frac{1}{\varepsilon}}} & \text{for} \quad d = 2 \\[2ex] \dfrac{|\Omega|}{4(a+b)D\tilde{r}} \dfrac{1 - 16ab\tilde{r}^2\left(\frac{1}{2\pi|a+\Delta'+b|} + O(1)\right)^2}{1 - \frac{8ab\tilde{r}}{a+b}\left(\frac{1}{2\pi|a+\Delta'+b|} + O(1)\right)} & \text{for} \quad d = 3, \end{cases} \tag{3}$$

as $a, b, \varepsilon, \delta, \Delta' \to 0$. Here $\tilde{r} = \tilde{r}(\Delta', \varepsilon, \delta)$ is a function of $\Delta', \varepsilon, \delta$ that varies monotonically between $\tilde{r}(0, 0, 0) \approx 0.6$ and $\tilde{r}(\Delta', 0, 0) \to 1$ as $\Delta' \to \infty$.

Equation (3) reduces to the single-window result (1) as $\delta \to 0$ or $b \to 0$. As the windows drift apart and $\Delta \gg \varepsilon, \delta, a, b$ the expression (3) gives $\lambda_1(\varepsilon, \delta) \approx \lambda_1(\varepsilon, 0) + \lambda_1(0, \delta)$, but as $\Delta \to 0$ (3) is a new result. It shows the nonlinear effect of clustering, which becomes more pronounced as the number of absorbing windows is large (see section 3). The term $O(1)$, which is due to the regular part of the Green–Neumann, expresses geometric properties of the absorbing boundary. Its analytical approximation seems much harder, but it can be estimated from numerical simulations of Brownian motion (see figures 3 and 4).

The mixed problem with a small Dirichlet cluster has many applications in cellular biology [12], because the MFPT to a Dirichlet cluster is the mean time for a diffusing messenger to hit a cluster of small targets. For example, this may be the mean time until a neurotransmitter,

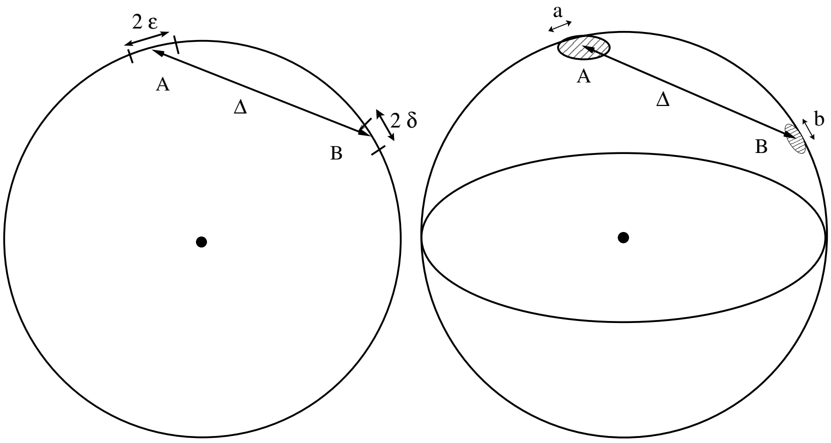

Figure 1. Schematic representation of a disk and a sphere with two holes on the boundary. In the plane, the arclengths of the holes are 2δ and 2ε, respectively, separated by Euclidean distance Δ, while in 3D, the radii of the holes are, respectively, a and b.

released into the synaptic cleft, binds to a receptor in a cluster on the boundary of a microdomain (the postsynaptic density [13]). Another example of clustered Dirichlet boundary is the clustering of transporters at the periphery of synapses and exchangers at dendrites [14]. Also, the asymptotic evaluation of the first eigenvalue is the first step toward the homogenization of the mixed problem for the diffusion equation with complicated reactive boundaries, which will be done separately.

2. Interaction between two small holes

2.1. Derivation of the Helmholtz integral equation

We consider Brownian motion in a bounded domain $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$, whose boundary $\partial\Omega$ is reflecting, except for two small absorbing {circular windows, $A \subset \partial\Omega$ and $B \subset \partial\Omega$, centered at $\mathbf{0}_A \in \partial\Omega$ and $\mathbf{0}_B \in \partial\Omega$, of sizes $(|A|/|\partial\Omega|)^{1/(d-1)} = \varepsilon$ and $(|B|/|\partial\Omega|)^{1/(d-1)} = \delta$, which are the radii of A and B, respectively, if the domain is normalized so that $\partial\Omega = \pi$. In contrast to the case of a single absorbing window, the mean time to absorption (MTA) does not diverge to infinity as the size of one of the windows decreases to zero. Thus the computation of the MTA differs from that for the single-window case given in [5]. Our purpose is to derive an asymptotic approximation to the probability flux through each window for $\varepsilon, \delta \ll 1$.

The MTA, $u(\boldsymbol{x}) = E[\tau|\boldsymbol{x}(0) = \boldsymbol{x}]$, is the solution of the mixed Neumann–Dirichlet boundary value problem (see for example [17])

$$D\Delta u(\boldsymbol{x}) = -1 \qquad \text{for} \quad \boldsymbol{x} \in \Omega \tag{4}$$

$$\frac{\partial u(\boldsymbol{x})}{\partial n} = 0 \qquad \text{for} \quad \boldsymbol{x} \in \partial\Omega - \partial\Omega_a \tag{5}$$

$$u(\boldsymbol{x}) = 0 \qquad \text{for} \quad \boldsymbol{x} \in \partial\Omega_a, \tag{6}$$

where D is the diffusion coefficient. We set

$$g(\boldsymbol{x}) = D\frac{\partial u(\boldsymbol{x})}{\partial n} \qquad \text{for} \quad \boldsymbol{x} \in \partial\Omega_a.$$

The fluxes are defined by

$$\Phi_A = -\int_A g(\boldsymbol{x})\,\mathrm{d}S_{\boldsymbol{x}}, \qquad \Phi_B = -\int_B g(\boldsymbol{x})\,\mathrm{d}S_{\boldsymbol{x}}.$$

To compute the fluxes, we first integrate equation (4) over the domain and we get

$$\Phi_A + \Phi_B = |\Omega|. \tag{7}$$

We then use the Neumann function, as in [5], to construct an asymptotic approximation to the solution $u(\boldsymbol{x})$ of the mixed boundary value problem (4)–(6). We assume, for convenience, that $D = 1$. The Neumann function $N(\boldsymbol{x}, \boldsymbol{\xi})$, as defined in [15], is the solution of the boundary value problem

$$D\Delta_{\boldsymbol{x}} N(\boldsymbol{x}, \boldsymbol{\xi}) = -\delta(\boldsymbol{x} - \boldsymbol{\xi}) \qquad \text{for} \quad \boldsymbol{x}, \boldsymbol{\xi} \in \Omega \tag{8}$$

$$D\frac{\partial N(\boldsymbol{x}, \boldsymbol{\xi}))}{\partial n(\boldsymbol{x})} = -\frac{1}{|\partial\Omega|} \qquad \text{for} \quad \boldsymbol{x} \in \partial\Omega, \boldsymbol{\xi} \in \Omega, \tag{9}$$

and is defined up to an additive constant. It has the form

$$N(\boldsymbol{x}, \boldsymbol{\xi}) = \begin{cases} \dfrac{1}{D\sigma_{d-1}}|\boldsymbol{x} - \boldsymbol{\xi}|^{-d+2} + v_S(\boldsymbol{x}, \boldsymbol{\xi}) & \text{for} \quad d > 2, \quad \boldsymbol{x}, \boldsymbol{\xi} \in \Omega \\ -\dfrac{1}{2D\pi}\log|\boldsymbol{x} - \boldsymbol{\xi}| + v_S(\boldsymbol{x}, \boldsymbol{\xi}) & \text{for} \quad d = 2, \quad \boldsymbol{x}, \boldsymbol{\xi} \in \Omega, \end{cases} \tag{10}$$

where $v_S(\boldsymbol{x}, \boldsymbol{\xi})$ is a regular harmonic function away from the boundary. For $d = 3$ it has a logarithmic singularity, which is proportional to the mean curvature of the boundary in the window. More specifically [16, p 247], for $|\boldsymbol{x}| = |\boldsymbol{\xi}| = R$, and $\angle(\boldsymbol{x}, \boldsymbol{\xi}) = \gamma$, the logarithmic part of the Neumann function for a sphere of radius R is

$$v_S(\boldsymbol{x}, \boldsymbol{\xi}) = -\frac{1}{4\pi DR}\log\left(\sin^2\frac{\gamma}{2} + \sin\frac{\gamma}{2}\right). \tag{11}$$

Therefore $v_S(\boldsymbol{x}, \boldsymbol{\xi})$ is bounded for $0 < \text{const} < \gamma < \pi$. The constant σ_{d-1} is the surface area of the unit sphere in $\mathbb{R}^d$.

To derive an integral representation of the solution, we multiply equation (4) by $N(\boldsymbol{x}, \boldsymbol{\xi})$, equation (8) by $u(\boldsymbol{x})$, integrate with respect to $\boldsymbol{x}$ over Ω, and use Green's formula to obtain the identity

$$D\oint_{\partial\Omega} N(\boldsymbol{x}, \boldsymbol{\xi})\frac{\partial u(\boldsymbol{x})}{\partial n}\,\mathrm{d}S_{\boldsymbol{x}} + \frac{1}{|\partial\Omega|}\oint_{\partial\Omega} u(\boldsymbol{x})\,\mathrm{d}S_{\boldsymbol{x}} = u(\boldsymbol{\xi}) - \int_\Omega N(\boldsymbol{x}, \boldsymbol{\xi})\,\mathrm{d}\boldsymbol{x}. \tag{12}$$

The second integral on the left-hand side of equation (12) is an additive constant, so we obtain the representation

$$u(\boldsymbol{\xi}) = \int_\Omega N(\boldsymbol{x}, \boldsymbol{\xi})\,\mathrm{d}\boldsymbol{x} + D\int_{\partial\Omega_a} N(\boldsymbol{x}, \boldsymbol{\xi})\frac{\partial u(\boldsymbol{x})}{\partial n}\,\mathrm{d}S_{\boldsymbol{x}} + C, \tag{13}$$

where

$$C = \frac{1}{|\partial\Omega|}\oint_{\partial\Omega} u(\boldsymbol{x})\,\mathrm{d}S_{\boldsymbol{x}}$$

is a constant to be determined from the boundary condition (6) and $dS_{\boldsymbol{x}}$ is a surface area element on $\partial\Omega_a$. We choose, respectively, $\boldsymbol{\xi} \in A$ and $\boldsymbol{\xi} \in B$, and using the boundary condition (6), we obtain the two equations

$$F(\boldsymbol{\xi}) = \int_A N(\boldsymbol{x}, \boldsymbol{\xi})g_A(\boldsymbol{x})\,\mathrm{d}S_{\boldsymbol{x}} + \int_B N(\boldsymbol{x}, \boldsymbol{\xi})g_B(\boldsymbol{x})\,\mathrm{d}S_{\boldsymbol{x}}, \qquad \text{for} \quad \boldsymbol{\xi} \in A \tag{14}$$

$$F(\boldsymbol{\xi}) = \int_A N(\boldsymbol{x},\boldsymbol{\xi}) g_A(\boldsymbol{x})\,\mathrm{d}S_{\boldsymbol{x}} + \int_B N(\boldsymbol{x},\boldsymbol{\xi}) g_B(\boldsymbol{x})\,\mathrm{d}S_{\boldsymbol{x}}, \qquad \text{for} \quad \boldsymbol{\xi} \in B, \tag{15}$$

where

$$F(\boldsymbol{\xi}) = -\left(\int_\Omega N(\boldsymbol{x},\boldsymbol{\xi})\,\mathrm{d}\boldsymbol{x} + C\right). \tag{16}$$

As discussed in [5], the first integral in equation (16) is a regular function of $\boldsymbol{\xi}$ on the boundary. Similar conditions have been derived in [18], for the case of diffusion though many holes in a plane separating two half-spaces.

The system of equation (14)–(15), called the Helmholtz integral equations [19], cannot be solved explicitly by using the methods presented in [1–4, 6]. Setting $r = |\boldsymbol{x} - \mathbf{0}_A|$, $(r' = |\boldsymbol{x} - \mathbf{0}_B|)$ for $d = 3$, we recall that for a single absorbing window A the fluxes calculated in *loc. cit.* are

$$g_A(\boldsymbol{x}) \sim \begin{cases} \dfrac{\tilde{g}_A f\left(\frac{r}{\varepsilon}\right)}{\sqrt{1-\frac{r^2}{\varepsilon^2}}} & \text{for} \quad \boldsymbol{x} \in A, \quad d = 2 \\ \dfrac{\tilde{g}_A}{\sqrt{1-\frac{r^2}{\varepsilon^2}}} & \text{for} \quad \boldsymbol{x} \in A, \quad d = 3. \end{cases} \tag{17}$$

For $d = 2$ the variables r and r' are the signed arclengths in A and B, measured from their centers $\mathbf{0}_A$ and $\mathbf{0}_B$, respectively; the function $f(\alpha)$ is a positive smooth function for $|\alpha| \leqslant 1$ such that $f(0) = 1$. Equations (17) in A and B are an approximate solution of (14), (15) for well separated A and B and constants $\tilde{g}_A, \tilde{g}_B$. They are only approximations, because the integral $\int_B N(\boldsymbol{x},\boldsymbol{\xi}) g_B(\boldsymbol{x})\,\mathrm{d}S_{\boldsymbol{x}}$ is not constant for $\boldsymbol{\xi} \in A$, though it is much smaller than $\int_A N(\boldsymbol{x},\boldsymbol{\xi}) g_A(\boldsymbol{x})\,\mathrm{d}S_{\boldsymbol{x}}$ there. If, however, A and B are not well separated (17) (in A and in B) is not even an approximate solution, because the integrals are of comparable orders of magnitude.

2.2. *The first eigenvalue for $d = 2$*

First, we estimate integral $\int_A N(r,\xi) g_A(r)\,\mathrm{d}r$ in case the two windows are identical (the general case is similar) for a planar problem. We set $\Delta = \varepsilon + \Delta' + \delta$ and recall that it was shown in [7] that the flux density through a single window of size 2ε is given by (17), where $f_\Delta(x)$ is a smooth positive even function for $-1 \leqslant x \leqslant 1$ and $f_\Delta(0) = 1$. The computations of [5, 7] show that the solution of the two windows problem (14)–(15) has the form

$$g_A(r) = \frac{\tilde{g}_A f_\Delta\left(\frac{r}{\varepsilon}\right)}{\sqrt{1-\frac{r^2}{\varepsilon^2}}} \qquad \text{for} \quad -\varepsilon \leqslant r \leqslant \varepsilon, \tag{18}$$

$$g_B(r) = \frac{\tilde{g}_B f_\Delta\left(\frac{r}{\varepsilon}\right)}{\sqrt{1-\frac{r^2}{\varepsilon^2}}} \qquad \text{for} \quad \varepsilon + \Delta' \leqslant r \leqslant 3\varepsilon + \Delta', \tag{19}$$

where $\tilde{g}_A$ and $\tilde{g}_B$ are constants and $f_\Delta(x)$ is a positive smooth function for $-1 \leqslant x \leqslant 1$, $1 + \frac{\Delta'}{\varepsilon} \leqslant x \leqslant 3 + \frac{\Delta'}{\varepsilon}$, such that $f_\Delta(0) = f_\Delta\left(2 + \frac{\Delta'}{\varepsilon}\right) = 1$. To estimate the solution of the mixed problem in A and B, we begin with

$$\int_{-\varepsilon}^{\varepsilon} N(r,\xi) g_A(r)\,\mathrm{d}r = \int_{-\varepsilon}^{\varepsilon} \frac{\tilde{g}_A\left(v_S(\boldsymbol{x},\boldsymbol{\xi}) - \frac{1}{D\pi}\log|r-\xi|\right)}{\sqrt{1-\frac{r^2}{\varepsilon^2}}} f_\Delta\left(\frac{r}{\varepsilon}\right)\mathrm{d}r.$$

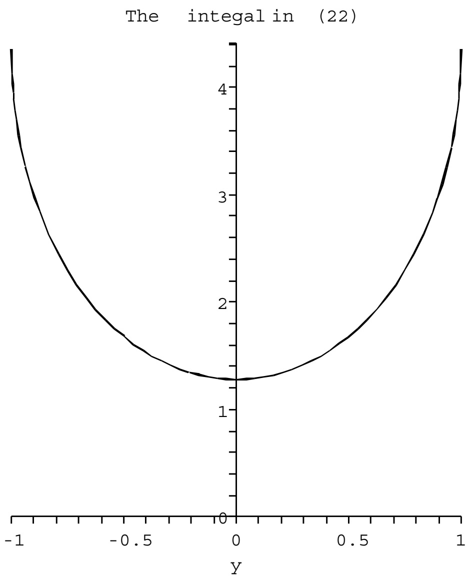

Figure 2. The function (23).

Scaling $r = \varepsilon x, \xi = \Delta' + 2\varepsilon + \varepsilon y$ and approximating $v_S(\boldsymbol{x}, \boldsymbol{\xi}) = v_S(\mathbf{0}_A, 2\varepsilon + \Delta') + O(\varepsilon)$, uniformly for all $\boldsymbol{x} \in A$, $\boldsymbol{\xi} \in B$, and $\Delta', \varepsilon < 1$, we obtain

$$\begin{aligned}
&\int_{-\varepsilon}^{\varepsilon} N(r, \xi) g_A(r)\,\mathrm{d}r \\
&\quad = -\frac{1}{D\pi}\varepsilon \tilde{g}_A \int_{-1}^{1} \frac{\left[\log(2\varepsilon + \Delta') - \pi v_S(\mathbf{0}_A, 2\varepsilon + \Delta') + O(\varepsilon) + \log\left|1 + \delta' \frac{x-y}{2}\right|\right]}{\sqrt{1 - x^2}} f_\Delta(x)\,\mathrm{d}x \\
&\quad = -\frac{1}{D\pi}\{\varepsilon \tilde{g}_A \alpha[\log(2\varepsilon + \Delta') - \pi v_S(\mathbf{0}_A, 2\varepsilon + \Delta') + O(\varepsilon)] + \varepsilon \tilde{g}_A \alpha(y)\},
\end{aligned} \tag{20}$$

where $\delta' = \frac{2\varepsilon}{2\varepsilon + \Delta'} < 1$ and

$$\alpha = \int_{-1}^{1} \frac{f_\Delta(x)\,\mathrm{d}x}{\sqrt{1 - x^2}}, \qquad \alpha(y) = \int_{-1}^{1} \log\left|1 + \delta' \frac{x - y}{2}\right| \frac{f_\Delta(x)\,\mathrm{d}x}{\sqrt{1 - x^2}}. \tag{21}$$

Using the inequality

$$\begin{aligned}
\left|\log\left|1 + \delta' \frac{x - y}{2}\right|\right| &= \left|-\sum_{n=1}^{\infty} (-1)^{n+1} \frac{(\delta')^n}{n} \left(\frac{x - y}{2}\right)^n\right| \leqslant \sum_{n=1}^{\infty} \frac{|x - y|^n}{n 2^n} \\
&= -\log\left|1 - \frac{|x - y|}{2}\right| \qquad \text{for} \quad |x|, |y| < 1
\end{aligned} \tag{22}$$

and the bound (see figure 2)

$$1.281\,286\,760 \leqslant \int_{-1}^{1} \log\left|1 - \frac{|x - y|}{2}\right| \frac{\mathrm{d}x}{\sqrt{1 - x^2}} \leqslant 4.355\,172\,182, \tag{23}$$

we obtain

$$0 < |\alpha(y)| \leqslant -\int_{-1}^{1} \log\left|1 - \frac{|x - y|}{2}\right| \frac{f_\Delta(x)\,\mathrm{d}x}{\sqrt{1 - x^2}} \leqslant 4.355\,172\,182 \max_{|x| \leqslant 1} f_\Delta(x),$$

so that the logarithmic term dominates the right-hand side of (20) for small and large values of Δ'.

Since $v_S(\boldsymbol{x},\boldsymbol{\xi})$ is a uniformly bounded function for $\boldsymbol{x}\in A$ and $\boldsymbol{\xi}\in B$, it follows from (20) that

$$\int_{-\varepsilon}^{\varepsilon} N(r,\xi)g_A(r)\,\mathrm{d}r=\varepsilon\alpha\tilde{g}_A[N(\mathbf{0}_A,\mathbf{0}_B)+O(1)]\qquad\text{for all}\quad \xi\geqslant\varepsilon, \tag{24}$$

and

$$N(\mathbf{0}_A,\mathbf{0}_B)=-\frac{1}{D\pi}\log(2\varepsilon+\Delta')+O(1),$$

where $O(1)$ is a uniformly bounded function of ε and Δ' for $0<\varepsilon<1$, $\Delta'\geqslant 0$. If the radii of A and B are ε and δ, respectively, then

$$N(\mathbf{0}_A,\mathbf{0}_B)=-\frac{1}{D\pi}\log(\varepsilon+\delta+\Delta')+O(1),$$

uniformly for $0<\varepsilon,\delta<1$, $\Delta'>0$. Using (24) and the computations [7], we get that

$$\int_A N(\boldsymbol{x},\boldsymbol{\xi})g_A(\boldsymbol{x})\,\mathrm{d}S_{\boldsymbol{x}}=\begin{cases}\dfrac{\alpha\varepsilon[-\log\varepsilon+O(1)]\tilde{g}_A}{D\pi} & \text{for}\quad \boldsymbol{\xi}\in A\\[2ex] -\dfrac{\varepsilon\alpha\tilde{g}_A\left[\log(\varepsilon+\delta+\Delta')+O(1)\right]}{D\pi} & \text{for}\quad \boldsymbol{\xi}\in B.\end{cases} \tag{25}$$

and

$$\int_B N(\boldsymbol{x},\boldsymbol{\xi})g_B(\boldsymbol{x})\,\mathrm{d}S_{\boldsymbol{x}}=\begin{cases}\dfrac{\beta\delta[-\log\delta+O(1)]\tilde{g}_B}{D\pi} & \text{for}\quad \boldsymbol{\xi}\in B\\[2ex] -\dfrac{\delta\beta\tilde{g}_B\left[\log(\varepsilon+\delta+\Delta')+O(1)\right]}{D\pi} & \text{for}\quad \boldsymbol{\xi}\in A,\end{cases} \tag{26}$$

where $O(1)$ is a uniformly bounded function for $0<\varepsilon,\delta<1$, $\Delta'>0$ and $\boldsymbol{\xi}\in A\cup B$.

For $\varepsilon\ll 1$ and $\delta\ll 1$ the logarithmic terms in the first lines of (25) and (26) are dominant, but not in the second lines, because the regular part of the Neumann function can be of the same order of magnitude as the leading term.

Using the boundary conditions (14) and estimates (25), (26), we see that to leading order

$$\frac{\alpha\varepsilon[\log\varepsilon+O(1)]\tilde{g}_A}{D\pi}+\frac{\delta\beta\tilde{g}_B[\log(\varepsilon+\delta+\Delta')+O(1)]}{D\pi}=C$$
$$\frac{\varepsilon\alpha\tilde{g}_A[\log(\varepsilon+\delta+\Delta')+O(1)]}{D\pi}+\frac{\beta\delta(\log\delta+O(1))\tilde{g}_B}{D\pi}=C$$

and

$$\begin{aligned}\frac{\alpha\tilde{g}_A}{\pi}&=-DC\frac{\log\frac{1}{\delta}-[\log(\varepsilon+\Delta'+\delta)+O(1)]}{\varepsilon\log\frac{1}{\varepsilon}\log\frac{1}{\delta}(1+\tilde{O}(1))-\varepsilon[\log(\varepsilon+\Delta'+\delta)+O(1)]^2}\\ \frac{\beta\tilde{g}_B}{\pi}&=-DC\frac{\log\frac{1}{\varepsilon}-[\log(\varepsilon+\Delta'+\delta)+O(1)]}{\delta\log\frac{1}{\delta}\log\frac{1}{\varepsilon}(1+\tilde{O}(1))-\delta[\log(\varepsilon+\Delta'+\delta)+O(1)]^2},\end{aligned} \tag{27}$$

where

$$\tilde{O}(1)=O\left(\frac{1}{\log\varepsilon}\right)+O\left(\frac{1}{\log\delta}\right).$$

Equations (18) and (21) give

$$\int_{-\varepsilon}^{\varepsilon} g_A(r)\,\mathrm{d}r=\int_{-\varepsilon}^{\varepsilon}\frac{\tilde{g}_A}{\sqrt{1-\frac{r^2}{\varepsilon^2}}}f_\Delta\left(\frac{r}{\varepsilon}\right)\,\mathrm{d}r=\alpha\varepsilon\tilde{g}_A,$$

so the flux condition (7) implies that for $d = 2$

$$\alpha\varepsilon\tilde{g}_A + \beta\delta\tilde{g}_B = -\frac{|\Omega|}{D}. \tag{28}$$

Therefore

$$C = \bar{\tau}_{A\cup B} = \frac{|\Omega|}{D\pi\left(\log\frac{1}{\varepsilon} + \log\frac{1}{\delta}\right)} \frac{\log\frac{1}{\delta}\log\frac{1}{\varepsilon}(1+\tilde{O}(1)) - [\log(\varepsilon+\Delta'+\delta) + O(1)]^2}{1 - 2\frac{[\log(\varepsilon+\Delta'+\delta)+O(1)]}{\log\frac{1}{\delta}+\log\frac{1}{\varepsilon}}}. \tag{29}$$

The constant $C = \bar{\tau}_{A\cup B}$ is the MTA in $A \cup B$. Formula (29) reduces to the single-window formula (1) in the limit $\delta \to 0$. If the two absorbing arcs are well separated, that is, if $\Delta' \gg \varepsilon, \delta$ for $\varepsilon, \delta \ll 1$, then $\log|\varepsilon + \Delta' + \delta| \ll \log\frac{1}{\delta}, \log\frac{1}{\varepsilon}$ and $\tilde{O}(1) = o(1)$, so the rate of absorption in $A \cup B$ reduces to

$$\frac{1}{\bar{\tau}_{A\cup B}} = \left(\frac{1}{\bar{\tau}_\varepsilon} + \frac{1}{\bar{\tau}_\delta}\right)(1 + o(1)) \qquad \text{for} \quad \varepsilon, \delta \ll 1, \tag{30}$$

which means that the two absorbing arcs do not interact. For example, when the two windows are well separated and if $\delta = \varepsilon$, equation (30) gives

$$\bar{\tau}_{A\cup B} = \frac{\bar{\tau}_\varepsilon}{2}, \tag{31}$$

where $\bar{\tau}_\varepsilon$ is the MTA in a single absorbing arc of length ε is given by (1).

For small Δ' formula (29) expresses the nonlinear interference between the two arcs. For example, if two arcs of length 2ε merge, formula gives, as Δ' shrinks to zero,

$$\bar{\tau}_{A\cup B} = \frac{|\Omega|}{\pi}\log\frac{1}{\varepsilon}(1 + o(1)) = \bar{\tau}_\varepsilon \qquad \text{for} \quad \log\frac{1}{\varepsilon} \gg 1, \tag{32}$$

which is identical to expansion (1), which applies to a single absorbing arc of the merged windows, that is, to a single arc of length 4ε. Note that the factor $1/2$ in (31) disappears in (32).

Brownian simulations for the empirical estimate of the MTA in $A \cup B$, as a function of the distance Δ' between the windows, give the results shown in figure 3. The order $O(1)$ term in the expansion (1) of the MTA shifts the curve parallel to the x-axis. Thus for two arcs with $\varepsilon = 0.03$, we get $\log(1/\varepsilon) \approx 3.6$ with the shift 1.3.

2.3. *The first eigenvalue for $d = 3$*

We consider two co-planar disks A and B centered at $\mathbf{0}_A = (0, 0)$ and $\mathbf{0}_B = (\Delta, 0)$, with radii ε and δ, respectively, and assume, as we may, that $\delta \leqslant \varepsilon$. We write $\boldsymbol{x} = \mathbf{0}_A + r(\cos\theta, \sin\theta)$, $\boldsymbol{x}' = \mathbf{0}_B + r'(\cos\theta', \sin\theta')$ and $\Delta = \varepsilon + \Delta' + \delta$, so that $|\mathbf{0}_A - \mathbf{0}_B| = \varepsilon + \Delta' + \delta$.

To estimate the solution of the Helmholtz equation (14), (15), we generalize (17) to A and B and write

$$\int_A N(\boldsymbol{x}, \boldsymbol{x}')g_A(\boldsymbol{x})\,\mathrm{d}S_{\boldsymbol{x}} = \int_0^{2\pi}\int_0^\varepsilon N(r', \theta', r, \theta)\frac{\tilde{g}_A f_\Delta\left(\frac{r}{\varepsilon}, \theta\right) r\,\mathrm{d}r}{\sqrt{1 - \frac{r^2}{\varepsilon^2}}}\,\mathrm{d}\theta, \tag{33}$$

where $\tilde{g}_A$ and $\tilde{g}_B$ are constants and $f_\Delta(\alpha, \theta)$ is a positive smooth function for $0 \leqslant \alpha \leqslant 1$, $0 \leqslant \theta < 2\pi$, such that $f_\Delta(0, \theta) = 1$. Using (10) and (11), we rewrite (33) for $\boldsymbol{x}' \in A$, as

$$\begin{aligned}
&\int_A N(\boldsymbol{x}, \boldsymbol{x}')g_A(\boldsymbol{x})\,\mathrm{d}S_{\boldsymbol{x}} \\
&\quad = \frac{1}{2D\pi}\int_0^{2\pi}\int_0^\varepsilon \frac{\tilde{g}_A f_\Delta\left(\frac{r}{\varepsilon}, \theta\right) r\,\mathrm{d}r\,\mathrm{d}\theta}{\sqrt{r^2 - 2rr'\cos(\theta - \theta') + r'^2}\sqrt{1 - \frac{r^2}{\varepsilon^2}}}(1 + o(1)) \\
&\quad = \frac{\varepsilon}{2\pi}\int_0^{2D\pi}\int_0^1 \frac{\tilde{g}_A f_\Delta(r, \theta) r\,\mathrm{d}r\,\mathrm{d}\theta}{\sqrt{r^2 - 2rr'\cos(\theta - \theta') + r'^2}\sqrt{1 - r^2}}(1 + o(1)) \qquad \text{as} \quad \varepsilon \to 0,
\end{aligned}$$

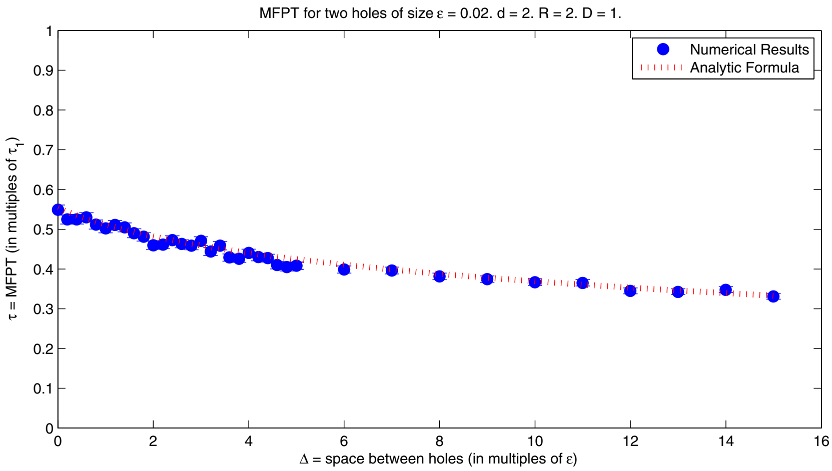

Figure 3. Normalized MTA as a function of the distance Δ' between two holes. We normalized by the volume and the y-axis represents the MTA normalized by τ_1 (MTA in a single hole). We choose $\varepsilon = \delta = 0.03$, the cell radius is $R = 5$ and τ_1 is computed according to formula 1. The contribution of the regular part of the Green function is estimated as 1.3 by a numerical fit.

where $o(1) = O(\varepsilon \log \varepsilon)$. First we integrate with respect to θ to obtain

$$\int_0^{2\pi} \frac{\mathrm{d}\theta}{\sqrt{r^2 - 2rr'\cos(\theta - \theta') + r'^2}} = \frac{4K\left(\frac{2\sqrt{rr'}}{r+r'}\right)}{r + r'},$$

where $K(\cdot)$ is the complete elliptic integral of the first kind. Expanding $f_\Delta(r, \theta)$ in Fourier series, we obtain an expansion of the outer integral in Fourier series in θ' with coefficients, which are integrals with respect to r of elliptic integrals of the variable $\frac{2\sqrt{rr'}}{r+r'}$ with weights of the form $\frac{P(r,r')}{\sqrt{1-r^2}}$, where $P(r, r')$ is a rational function. We approximate the Fourier series by the first term and since the integral is weakly dependent on r', we approximate it by its value at $r' = 0$. A direct evaluation gives that

$$\int_A \frac{g_A(\boldsymbol{x})}{|\boldsymbol{x} - \boldsymbol{x}'|}\,\mathrm{d}S_{\boldsymbol{x}} = \frac{1}{2\pi}\varepsilon \tilde{g}_A \tilde{\alpha}(1 + o(1)) \qquad \text{for} \quad \boldsymbol{x}' \in A, \quad \varepsilon < 1, \tag{34}$$

where

$$\tilde{\alpha} = \int_0^{2\pi} \mathrm{d}\theta \int_0^1 \frac{f_\Delta(x, \theta)\,\mathrm{d}x}{\sqrt{1 - x^2}}.$$

Equation (34) is a consequence of the expansion in [6] and the expansion of the Neumann function (10).

For $\boldsymbol{x}' \in B$, we scale $\boldsymbol{x} = \mathbf{0}_A + \varepsilon\boldsymbol{\eta}$, $\boldsymbol{x}' = \mathbf{0}_A + (\Delta + 2\varepsilon)\boldsymbol{u} + \varepsilon\boldsymbol{\xi}$, where the line between the centers $\mathbf{0}_A$ and $\mathbf{0}_B$ is spanned by the unit vector $\boldsymbol{u}$, and we write

$$\begin{aligned}\frac{1}{|\boldsymbol{x} - \boldsymbol{x}'|} &= [(\Delta + 2\varepsilon)^2 + 2\varepsilon^2 + 2\varepsilon(\Delta + 2\varepsilon)(\boldsymbol{\eta}, \boldsymbol{u}) + 2\varepsilon(\Delta + 2\varepsilon)(\boldsymbol{\xi}, \boldsymbol{u}) + 2\varepsilon^2(\boldsymbol{\eta}, \boldsymbol{\xi})]^{-1/2} \\ &= \frac{1}{(\Delta + 2\varepsilon)}\left[1 - \frac{\varepsilon}{(\Delta + 2\varepsilon)}((\boldsymbol{\xi}, \boldsymbol{u}) + (\boldsymbol{\eta}, \boldsymbol{u})) + O\left(\frac{2\varepsilon^2}{(\Delta + 2\varepsilon)^2}\right)\right].\end{aligned} \tag{35}$$

We obtain

$$\int_A N(\boldsymbol{x},\boldsymbol{x}')g_A(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = \frac{\varepsilon^2\tilde{g}_A(1+o(1))}{2D\pi(\Delta+2\varepsilon)}\int_0^{2\pi}\mathrm{d}\theta \times\int_0^1\left\{1-\frac{\varepsilon}{(\Delta+2\varepsilon)}[(\boldsymbol{\xi},\boldsymbol{u})+(\boldsymbol{\eta},\boldsymbol{u})]+O\left(\frac{2\varepsilon^2}{(\Delta+2\varepsilon)^2}\right)\right\}\frac{f_\Delta(z,\theta)z\,\mathrm{d}z}{\sqrt{1-z^2}},$$

where $(\boldsymbol{\eta},\boldsymbol{u})$ is a linear function of z. Expanding in Fourier series as above, we set $\boldsymbol{\xi}=\boldsymbol{0}$ and obtain the approximation

$$\int_A N(\boldsymbol{x},\boldsymbol{x}')g_A(\boldsymbol{x})\,\mathrm{d}S_{\boldsymbol{x}} \approx \begin{cases}\dfrac{\tilde{\alpha}\varepsilon\tilde{g}_A(1+o(1))}{2D\pi} & \text{for } \boldsymbol{x}'\in A,\quad \varepsilon<1\\ \varepsilon^2\tilde{g}_A\alpha[N(\boldsymbol{0}_A,\boldsymbol{0}_B)+O(1)] & \text{for } \boldsymbol{x}'\in B,\quad \varepsilon<1,\end{cases} \tag{36}$$

where

$$\alpha=\int_0^{2\pi}\mathrm{d}\theta\int_0^1\left\{1-\frac{\varepsilon}{(\Delta+2\varepsilon)}(\boldsymbol{\eta},\boldsymbol{u})+O\left(\frac{2\varepsilon^2}{(\Delta+2\varepsilon)^2}\right)\right\}\frac{f_\Delta(z,\theta)z\,\mathrm{d}z}{\sqrt{1-z^2}}. \tag{37}$$

Note that α depends weakly on ε and is evaluated below. Similar analysis of the integral over B gives

$$\int_B N(\boldsymbol{x},\boldsymbol{x}')g_B(\boldsymbol{x})\,\mathrm{d}S_{\boldsymbol{x}} \approx \begin{cases}\dfrac{\tilde{\beta}\delta\tilde{g}_B(1+o(1))}{2D\pi} & \text{for } \boldsymbol{x}'\in B,\quad \delta<1\\ \delta^2\tilde{g}_B\beta\,[N(\boldsymbol{0}_A,\boldsymbol{0}_B)+O(1)] & \text{for } \boldsymbol{x}'\in A,\quad \delta<1,\end{cases} \tag{38}$$

where β and $\tilde{\beta}$ are analogous to α and $\tilde{\alpha}$, respectively.

As A and B shrink the constant C diverges while $\int_\Omega N(\boldsymbol{x},\boldsymbol{x}')\,\mathrm{d}\boldsymbol{x}$ stays bounded. Therefore we can approximate equations (14) by

$$\begin{aligned} F(\boldsymbol{0}_A)&\approx -C=\frac{\tilde{\alpha}\varepsilon\tilde{g}_A(1+o(1))}{2D\pi}+\delta^2\tilde{g}_B\beta\,[N(\boldsymbol{0}_A,\boldsymbol{0}_B)+O(1)]\\ F(\boldsymbol{0}_B)&\approx -C=\frac{\tilde{\beta}\delta\tilde{g}_B(1+o(1))}{2D\pi}+\varepsilon^2\tilde{g}_A\alpha\,[N(\boldsymbol{0}_A,\boldsymbol{0}_B)+O(1)]\,. \end{aligned} \tag{39}$$

Thus, setting $R=\alpha/\tilde{\alpha}=\beta/\tilde{\beta}$, we write (39) as

$$\begin{aligned} \alpha\tilde{g}_A(1+o(1))&=C\frac{-\frac{1}{2D\pi R}+\delta\,[N(\boldsymbol{0}_A,\boldsymbol{0}_B)+O(1)]}{\frac{\varepsilon}{4\pi^2D^2R^2}-\delta\varepsilon^2\,[N(\boldsymbol{0}_A,\boldsymbol{0}_B)+O(1)]^2}\\ \beta\tilde{g}_B(1+o(1))&=C\frac{-\frac{1}{2D\pi R}+\varepsilon\,[N(\boldsymbol{0}_A,\boldsymbol{0}_B)+O(1)]}{\frac{\delta}{4\pi^2D^2R^2}-\varepsilon\delta^2\,[N(\boldsymbol{0}_A,\boldsymbol{0}_B)+O(1)]^2)}. \end{aligned}$$

To determine the value of the constant C, we use the flux integrals

$$\begin{aligned} &\int_A g_A(\boldsymbol{x})\,\mathrm{d}S_x=\varepsilon^2\alpha\tilde{g}_A(1+o(1)),\\ &\int_b g_B(\boldsymbol{x})\,\mathrm{d}S_x=\delta^2\beta\tilde{g}_B(1+o(1))\qquad \text{as}\quad \varepsilon,\delta\to 0. \end{aligned}$$

Using the flux condition (7) for two holes of radii ε and δ, we obtain

$$\varepsilon^2\alpha\tilde{g}_A+\delta^2\beta\tilde{g}_B=-\frac{|\Omega|}{D},$$

hence

$$C=\bar{\tau}_{A\cup B}=\frac{|\Omega|}{2D\pi R(\varepsilon+\delta)}\frac{1-4\pi^2D^2R^2\varepsilon\delta N(\boldsymbol{0}_A,\boldsymbol{0}_B)+O(1))^2}{1-4\pi DR\frac{\varepsilon\delta[N(\boldsymbol{0}_A,\boldsymbol{0}_B)+O(1)]}{\varepsilon+\delta}}. \tag{40}$$

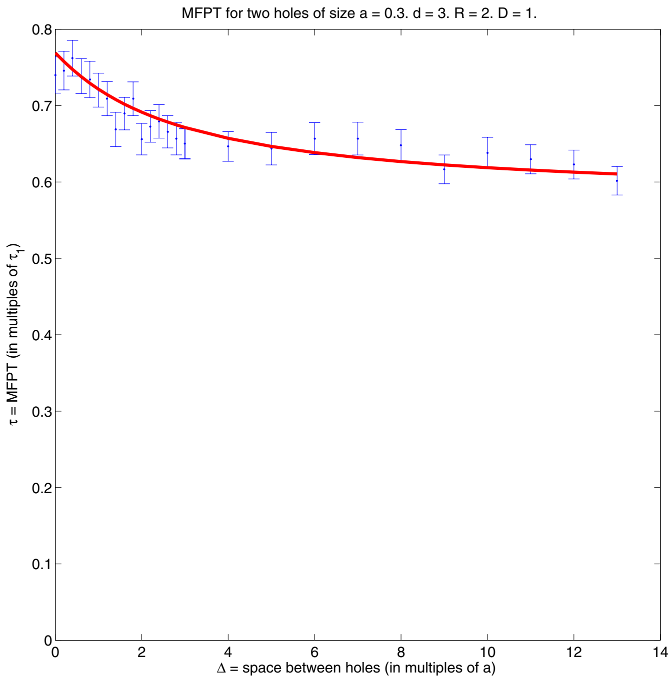

Figure 4. Normalized MTA as a function of the distance Δ between two holes. The MTA is normalized by the MTA in a single hole. The values of the parameters are $\varepsilon = \delta = 0.3$, the radius is $R = 2$, and τ_1 is computed from (1). To fit to Brownian simulations, the value $a_2 \approx 0.6$ and the regular part of the Green–Neumann function $O(1) = 0.4$ were chosen.

When the distance between the centers $\mathbf{0}_A$ and $\mathbf{0}_B$ is $\Delta = \varepsilon + \Delta' + \delta$, (10) and (11) give

$$N(\mathbf{0}_A, \mathbf{0}_B) = \frac{1}{2\pi D|\varepsilon + \Delta' + \delta|}[1 + O(\rho)], \tag{41}$$

where $\rho = \min\left(1, |\varepsilon+\Delta'+\delta| \log \frac{1}{|\varepsilon+\Delta'+\delta|}\right)$. The factor of 2π in the denominator of (41) comes from the coalescence of the image charges in Neumann's function on the boundary. Setting $\tilde{r} = DR\pi/2$, we write

$$\bar{\tau}_{A\cup B} = \frac{|\Omega|}{4(\varepsilon+\delta)\tilde{r}} \frac{1 - 16r^2\delta\varepsilon\left[\frac{1}{2\pi|\varepsilon+\Delta'+\delta|}(1+O(\rho))\right]^2}{1 - \frac{8\delta\varepsilon\tilde{r}}{\varepsilon+\delta}\frac{1}{2\pi|\varepsilon+\Delta'+\delta|}(1+O(\rho))}. \tag{42}$$

For a fixed D, the parameter r depends on Δ', δ and ε so we write $\tilde{r} = \tilde{r}(\Delta', \varepsilon, \delta)$. If Δ' is large, then $f_\Delta(r, \theta) = \text{const}$, so $\lim_{\Delta'\to\infty} \tilde{r}(\Delta', \varepsilon, \delta) = 1$. For $\delta, \varepsilon, \Delta' \to 0$, we determine the value of $\tilde{r}(0, \varepsilon, \delta)$ by fitting to numerical simulations of Brownian motion in a sphere with two tangent circular holes.

To test the range of validity of formula (40), we ran Brownian simulations in a three-dimensional ball of radius $R = 2$, containing two small absorbing caps of radius $a = 0.3$ on the boundary. In figure 4, we plot the MTA as a function of the distance Δ' between holes. For fixed ε and δ, we use a first-order Padé approximation to fit r as a function of Δ',

$$\tilde{r}(\Delta, \varepsilon, \delta) = a_1 \frac{\Delta'}{1+\Delta'} + a_2 \frac{1}{1+\Delta'}. \tag{43}$$

The fit at $\Delta' \to \infty$ gives $a_1 = 1$. Fitting to Brownian simulations, we find that the $O(1)$ correction in (40) is 0.4 and $a_2 \approx 0.6$. The results are represented in figure 4.

Because $\tilde{r}$ stays close to 1, we simplify (42) to

$$\bar{\tau}_{A\cup B} \approx \frac{|\Omega|}{4(\varepsilon+\delta)} \frac{1 - 16\delta\varepsilon\left[\frac{1}{2\pi|\varepsilon+\Delta'+\delta|}(1+O(\rho))\right]^2}{1 - \frac{8\delta\varepsilon}{\varepsilon+\delta}\frac{1}{2\pi|\varepsilon+\Delta'+\delta|}(1+O(\rho))}. \tag{44}$$

Formula (44) reduces to the single-window formula (1) if $\delta \to 0$. However, for two touching identical absorbing geodesic disks of radius $\varepsilon/2$ (i.e., $\Delta = 0$),

$$\bar{\tau}_{A\cup B} \approx \frac{|\Omega|}{4\varepsilon\tilde{r}}\left(1 + \frac{\tilde{r}}{\pi}\right), \qquad (\tilde{r} \approx 0.6). \tag{45}$$

Note that (45) is not the MFPT to a single circular absorbing window of double the area of a single window of radius $\varepsilon/2$ (which is $|\Omega|/2\sqrt{2}\varepsilon$). A more striking result is obtained by moving the windows apart (without changing the radius). The ratio between the MTAs (for $\Delta = 0$ and $\Delta = \infty$) is $\frac{1}{\tilde{r}}\left(1 + \frac{\tilde{r}}{\pi}\right) \approx 1.98$. This means that clustering may decrease the first eigenvalue (the flux) by about 50%.

3. Clusters of many holes

We now extend the above analysis to larger clusters. We consider clusters that are ensembles of circular Dirichlet windows $A_i \subset \partial\Omega$, $(i = 1, 2, \ldots, M)$, such that each has a neighbor separated from it by a distance comparable to their radii. The condition that the solution $u(\boldsymbol{x})$ of equation (12) vanishes on $\partial\Omega_a$ is

$$F(\boldsymbol{\xi}) = \sum_{i=1}^{M} \int_{A_i} N(\boldsymbol{x}, \boldsymbol{\xi}) g_i(\boldsymbol{x})\, \mathrm{d}S_{\boldsymbol{x}}, \qquad \text{for all} \quad \boldsymbol{\xi} \in \partial\Omega_a = \bigcup_{i=1}^{M} A_i, \tag{46}$$

where $F(\boldsymbol{\xi})$ is defined in (16) and the flux though window A_i is $g_i(\boldsymbol{x})$. Summing the contributions of all fluxes (7), we get

$$\sum_{i=1}^{M} \int_{A_i} g_i(\boldsymbol{x})\, \mathrm{d}S_{\boldsymbol{x}} = -|\Omega|. \tag{47}$$

Each circular window A_i is centered at $\mathbf{0}_i$ and has radius ε_i. As above, we fix the parameters $r_{i,j} = r(\Delta_{i,j}, \varepsilon_1, \ldots, \varepsilon_M)$ at $r_{i,j} = 1$. Using the explicit computation in dimension 3, given by formula (38) for $\boldsymbol{\xi} = \boldsymbol{\xi}_j \in A_j$, we obtain the system of $M + 1$ linear equations for the unknown $\tilde{g}_i$ and C,

$$\frac{\pi}{2}\varepsilon_j \tilde{g}_j + \sum_{i\neq j}^{M} 2\pi\varepsilon_i^2 \tilde{g}_i N(i, j) = F(\boldsymbol{\xi}_j) \approx -C, \qquad j = 1, \ldots, M, \tag{48}$$

$$2\pi \sum_{i=1}^{M} \varepsilon_i^2 \tilde{g}_i = -|\Omega|, \tag{49}$$

where $N(i, j) = N(\mathbf{0}_i, \mathbf{0}_j)$. Equation (49) is the flux condition. We assume that all radii ε_i can be scaled by $\varepsilon_i = \varepsilon\tilde{\varepsilon}_i$, where $\varepsilon = \min_{1\leqslant i\leqslant M} \varepsilon_i \ll 1$. Then, for windows separated by distance $\Delta_{i,j}$,

$$\max_{i,j} 2\varepsilon N(i, j) = \max_{i,j} \frac{1}{\pi\left(\tilde{\varepsilon}_i + \tilde{\varepsilon}_j + \frac{\Delta_{i,j}}{\varepsilon}\right)}(1 + o(1)) < 1. \tag{50}$$

Scaling $G_j = \pi\varepsilon_j\tilde{g}_j/C$, we write the symmetric matrix of the system (48) (with $1/2$ on the diagonal) as

$$\boldsymbol{M} = \begin{pmatrix} 1/2 & 2\tilde{\varepsilon}_2 N(1,2) & \cdots & 2\tilde{\varepsilon}_M N(1,M) \\ 2\tilde{\varepsilon}_2 N(1,2) & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ 2\tilde{\varepsilon}_M N(1,M) & \cdots & \cdots & 1/2 \end{pmatrix}. \tag{51}$$

We decompose $\boldsymbol{M}$ as

$$\boldsymbol{M} = \tfrac{1}{2}\boldsymbol{I}_M + \varepsilon\boldsymbol{A},$$

where $\boldsymbol{I}_M$ is the identity matrix and $\boldsymbol{A}$ contains off-diagonal terms. Writing $\mathbf{1}_M$ (resp. $\tilde{\boldsymbol{G}}_M$) for a vector of 1 (resp. G_j), (48) becomes

$$\left(\tfrac{1}{2}\boldsymbol{I}_M + \varepsilon\boldsymbol{A}\right)\tilde{\boldsymbol{G}}_M = -\mathbf{1}_M \tag{52}$$

and can be inverted as the convergent series

$$\tilde{\boldsymbol{G}}_M = -2\sum_{k=0}^{\infty}(-2\varepsilon\boldsymbol{A})^k\mathbf{1}_M. \tag{53}$$

All terms can contribute to the sum, because $\varepsilon N(i,j)$ can be of order 1. The interaction of the cluster with window j is given by

$$G_j = -2 - 2\sum_{k=0}^{\infty}(-2\varepsilon)^k \sum_{i_1,\ldots,i_k} N(j,i_1)N(i_1,i_2)\cdots N(i_{k-1},i_k), \tag{54}$$

where the sum is over all non-diagonal pairs (not all i_k are different). The nonlinearity depends on the number of windows. In the first approximation,

$$\tilde{\boldsymbol{G}}_M \approx -2(\boldsymbol{I}_M - 2\varepsilon\boldsymbol{A})\mathbf{1}_M \tag{55}$$

and

$$\pi\frac{\varepsilon_j\tilde{g}_j}{C} = G_j = -2\left(1 - 2\varepsilon\sum_{i\neq j}\tilde{\varepsilon}_i N(i,j)\right). \tag{56}$$

Using condition (49), we obtain for the constant C the equation

$$-4C\sum_{i=1}^{M}\varepsilon_i\left(1 - 2\sum_{i\neq j}\varepsilon_i N(i,j)\right) = -|\Omega|,$$

so

$$C = \bar{\tau}_{\bigcup A_i} \approx \frac{|\Omega|}{4D}\frac{1}{\sum_{i=1}^{M}\varepsilon_i\left(1 - 2\sum_{i\neq j}\varepsilon_j N(i,j)\right)} \tag{57}$$

and the flux coefficients are

$$\tilde{g}_j \approx \frac{|\Omega|}{2D\pi\varepsilon_i}\frac{\left(1 - 2\sum_{i\neq j}\varepsilon_i N(i,j)\right)}{\sum_{i=1}^{M}\varepsilon_i\left(1 - 2\sum_{i\neq j}\varepsilon_j N(i,j)\right)}. \tag{58}$$

The total flux through hole A_i is given by

$$\Phi_i = \int_A g_i(S')\,\mathrm{d}S' = 2\pi\varepsilon_i^2\tilde{g}_i \approx \frac{|\Omega|}{D}\frac{\varepsilon_i\left(1 - 2\sum_{i\neq j}\varepsilon_j N(i,j)\right)}{\sum_{i=1}^{M}\varepsilon_i\left(1 - 2\sum_{i\neq j}\varepsilon_j N(i,j)\right)}. \tag{59}$$

In the case of maximal packing, around a disk of radius ε surrounded by touching windows of the same radius, the maximum number of touching disk is $M = 6$ in dimension 2. Thus at

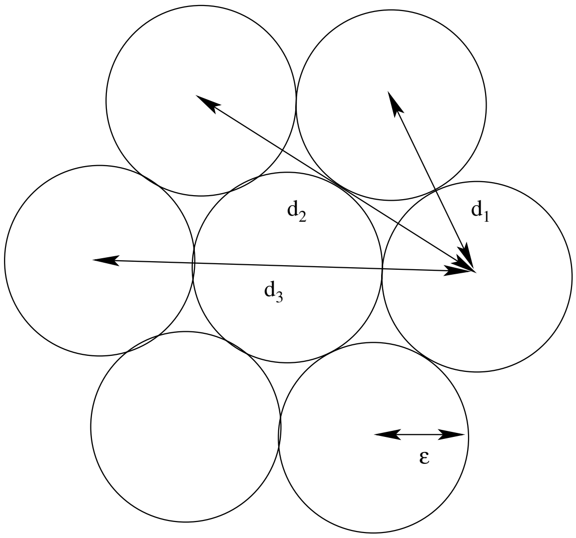

Figure 5. Cluster of six Dirichlet windows packed at the density limit. Each disk has a radius ε and using the three distances d_1, d_2, d_3, we can use formula (57) to estimate the mean first passage time to the cluster.

order 1, using formula (57), we can estimate the MTA in the cluster of disks (see figure 5). Using the symmetries of order 3 in figure 5, formula (57) for this specific cluster is

$$\bar{\tau}_{fl} \approx \frac{|\Omega|}{4DM\varepsilon}\left(1 + 2\varepsilon\frac{\sum_{i\neq j} N(i,j)}{M} + o(\varepsilon)\right). \tag{60}$$

We label 0 the centered window and the other are labeled clockwise from 1 to 6. The contribution to the sum of the central window is $MN(0,1)$ while that of the non-centered windows is to leading order $M(3N(0,1) + 2N(1,3) + N(1,4)$, where $N(0,1) = \frac{1}{2\pi Dd_1}$, $N(1,3) = \frac{1}{2D\pi d_2}$, $N(1,4) = \frac{1}{2\pi Dd_3}$ and $d_1 = 2\varepsilon$, $d_1 = 2\sqrt{3}\varepsilon$, $d_3 = 4\varepsilon$. Thus

$$\begin{aligned}\bar{\tau}_{fl} &\approx \frac{|\Omega|}{4DM\varepsilon}\{1 + 2\varepsilon[4N(0,1) + 2N(1,3) + N(1,4) + O(1)]\}\\ &= \frac{|\Omega|}{4DM\varepsilon}\left[1 + \frac{2}{\pi}\left(1 + \frac{1}{2\sqrt{3}} + \frac{1}{8}\right)\right] + O(1).\end{aligned}$$

4. Discussion

We derived and asymptotic approximation to the first eigenvalue of mixed Dirichlet–Neumann boundary value problem for the Laplace equation under the assumption that the Dirichlet boundary consists of a finite number of small arcs in dimension 2 or small disks in dimension 3. We analyze the effect of clustering of the Dirichlet windows on the first eigenvalue and found that clustering may affect the first eigenvalue significantly. In the application to Brownian motion, we found the effect on the MTA and obtained an explicit approximation to the probability flux of escaping Brownian trajectories. We still do not know what are the geometrical features of the Dirichlet windows, which determine the overall MTA.

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